An introduction to measurement uncertainty

Ian Smith
25 January 2018

Data Science
National Physical Laboratory, UK

Outline

- Introduction
- Main stages of uncertainty evaluation
- Formulation example
- GUM uncertainty framework
- Constructing an uncertainty budget
- Further concepts
- Summary
- Useful links

Introduction

- Purpose of measurement is to provide information about a quantity of interest (the measurand)
- No measurement is perfect
- Information about the measurand is incomplete
- As a result there is uncertainty in knowing the value of the measurand
- Only possible to state the probability that the value lies within a given interval
- Emphasise strongly probabilistic basis for uncertainty evaluation

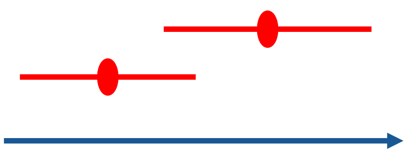
Where do uncertainties come from?

- The measuring system
  - Calibration, resolution, drift, ageing, …
- The item being measured
  - Stability, homogeneity, …
- The operator
- The environment
  - Temperature, air pressure, humidity, …

Why are uncertainties important?

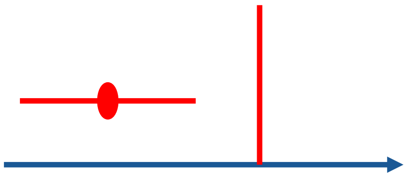
- To quantify the quality of a measured value

- To compare different measured values
  E.g., from different measuring systems



- To compare a measured value with theory

- To compare a measured value with a tolerance
  E.g., in conformity assessment



Definitions

National Physical Laboratory

- “a non-negative parameter characterizing the dispersion of quantity values being attributed to a measurand, based on the information used” (VIM)

- “a parameter, associated with the result of a measurement, that characterizes the dispersion of the values that could reasonably be attributed to the measurand” (GUM)

- Different ways of reporting measurement uncertainty
  - Standard uncertainty
  - Expanded uncertainty
  - Coverage interval for a stated coverage probability

Reporting a measurement result

NPL National Physical Laboratory

- E.g., as an interval with the uncertainty stated for a 95 % level of confidence ("95 % coverage interval")

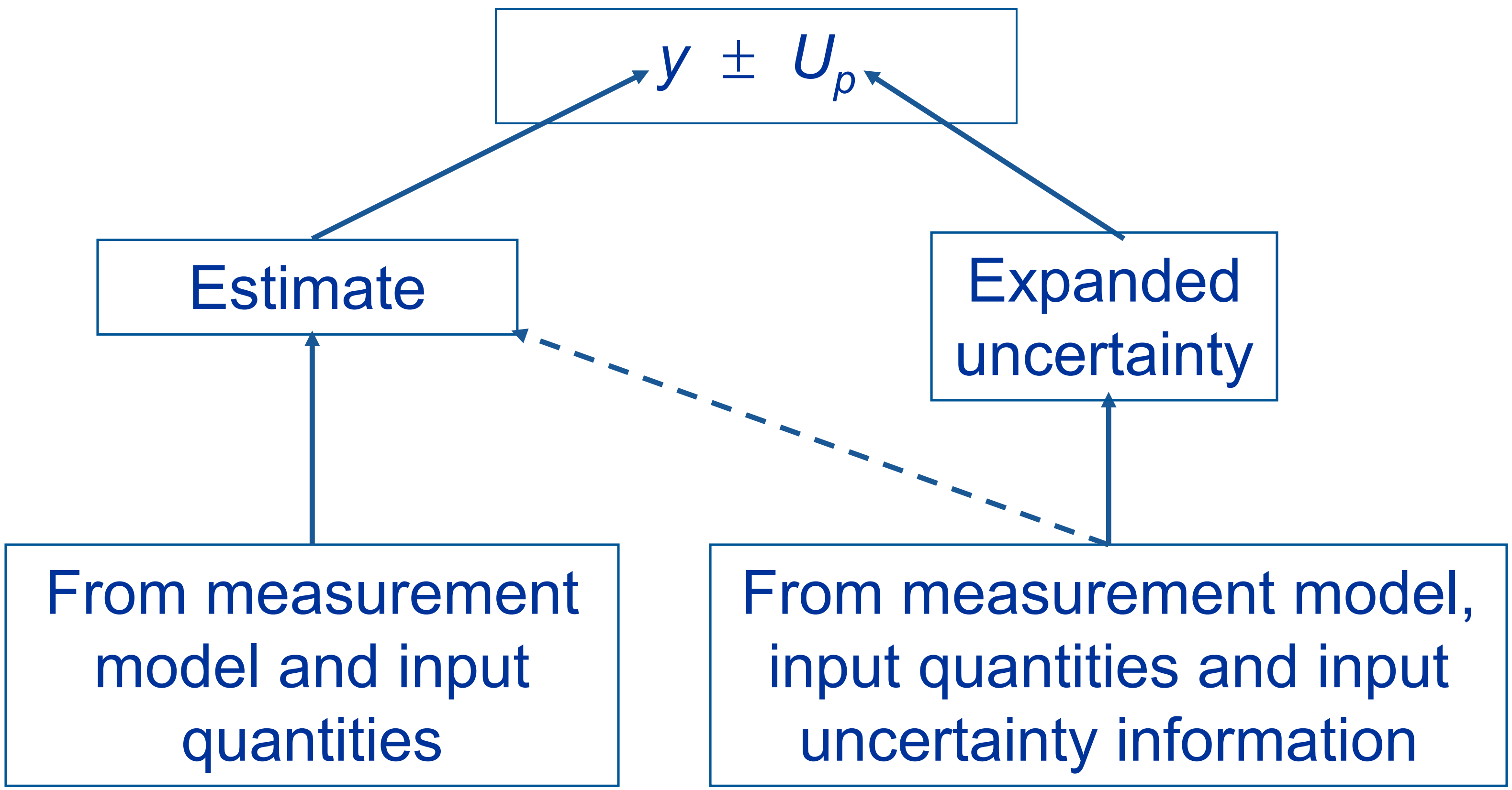
1.000 34 kg ± 0.000 08 kg

1.000 342 kg ± 0.000 083 kg

Number of digits after point here dictates number here

- Relative standard uncertainties are also used

Reporting a measurement result



Guide to the expression of uncertainty in measurement (GUM)

- Primary document regarding measurement uncertainty
- GUM provides a framework for uncertainty evaluation
- “Other analytic or numerical methods” (GUM Clause G.1.5) where appropriate
- A measurement model is central to consideration
- Probability is central to consideration

Main stages of uncertainty evaluation

- Main stages of uncertainty evaluation
  - Formulation stage (metrological)
  - Calculation stages (computational) comprising propagation and summarising

Formulation stage

- Decide what is the measurand (output quantity)
  - the quantity intended to be measured Y
- Decide what are the input quantities on which the measurand depends $X_1, X_2, \ldots, X_N$
  - indication quantities, applied corrections, calibration quantities (of artefacts, instruments), etc.
- Decide the relationship between the measurand and the input quantities
  - the measurement model $Y = f(X_1, X_2, \ldots, X_N)$
- Gather information/knowledge about the input quantities

Formulation stage

- Metrologist derives measurement model relating input and output quantities
  - some general principles
  - usually discipline-specific

- Metrologist uses available knowledge to characterise input quantities by probability density functions
  - approaches available

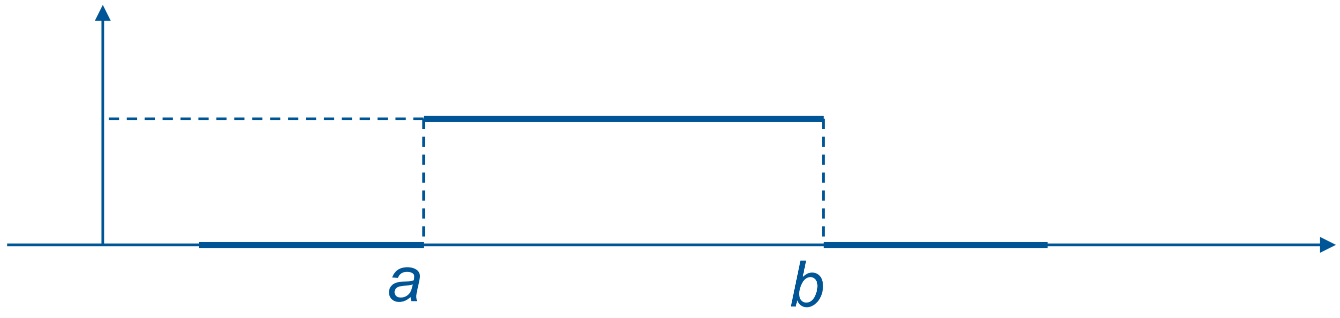
Probability density functions (PDFs)

- Value of a quantity generally not known exactly
- PDF summarises what is known about a quantity in a probabilistic sense
- Most commonly encountered PDFs
  - Rectangular (uniform) distribution
  - Gaussian (normal) distribution
  - *t*-distribution
- Many others

Rectangular (uniform) distribution

- Value of quantity expected to lie within a certain interval with equal probability
- Defined in terms of end-points a and b of the interval

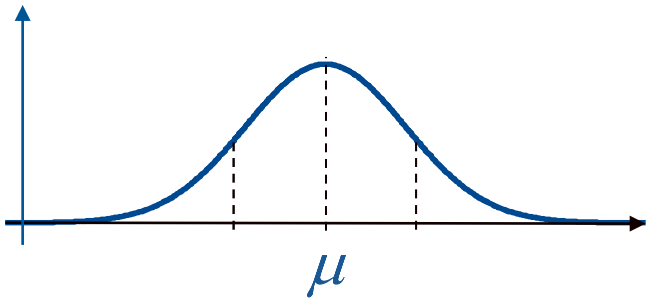
  R(*a*, *b*)



Gaussian (normal) distribution

- Widely encountered distribution
- Value of quantity has high probability of lying close to a "central" value and low probability of lying far from "central" value
- Defined in terms of mean μ and standard deviation σ

  N(μ, σ^2)

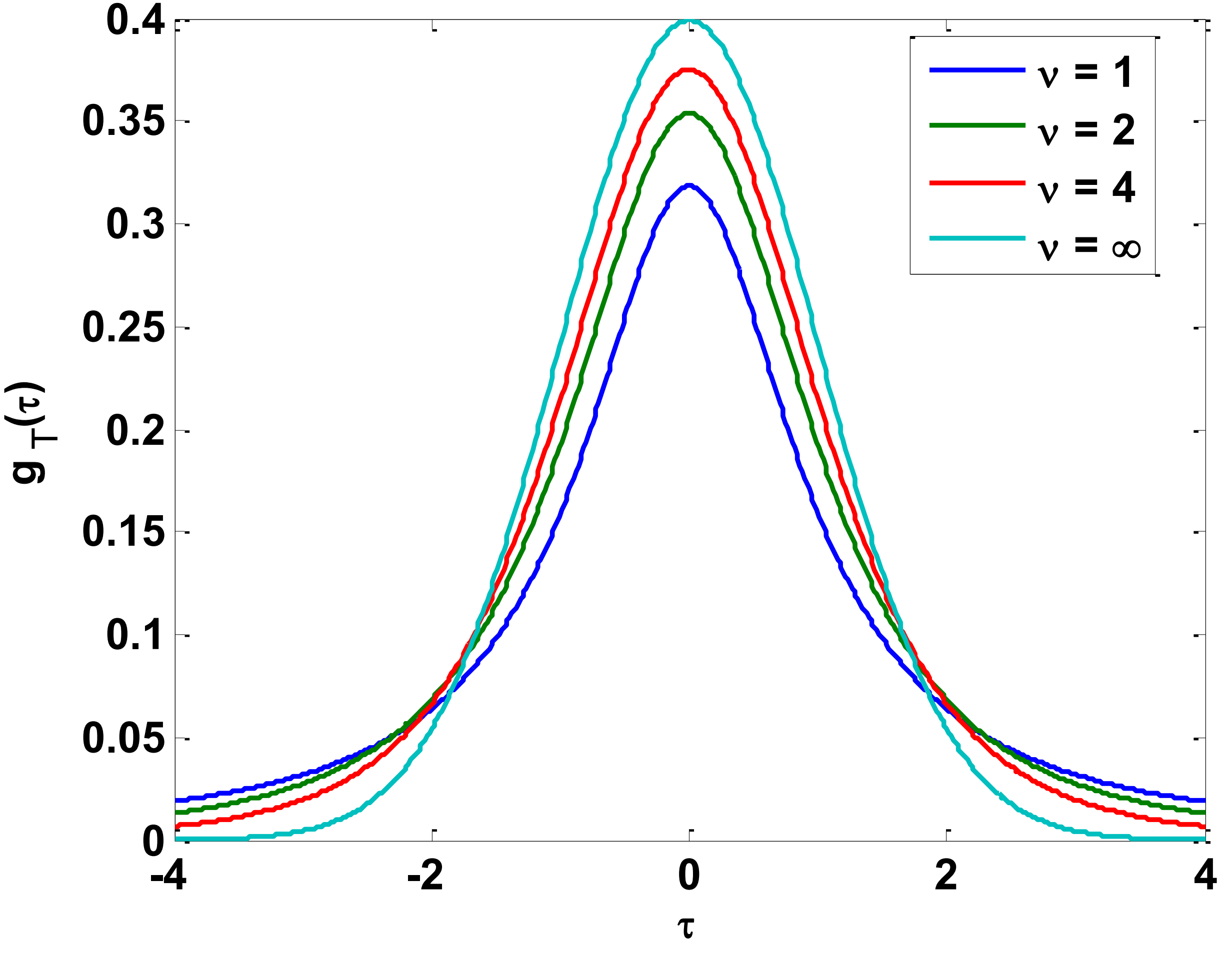


t-distribution

- Arises when information about a quantity takes the form of repeated indication values drawn from a Gaussian distribution
- Defined in terms of "central" value μ, scale s and degrees of freedom ν

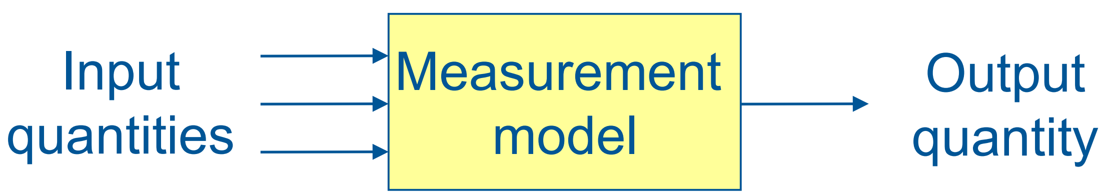
$$t_\nu(\mu, s^2)$$

t-distribution



Measurement model

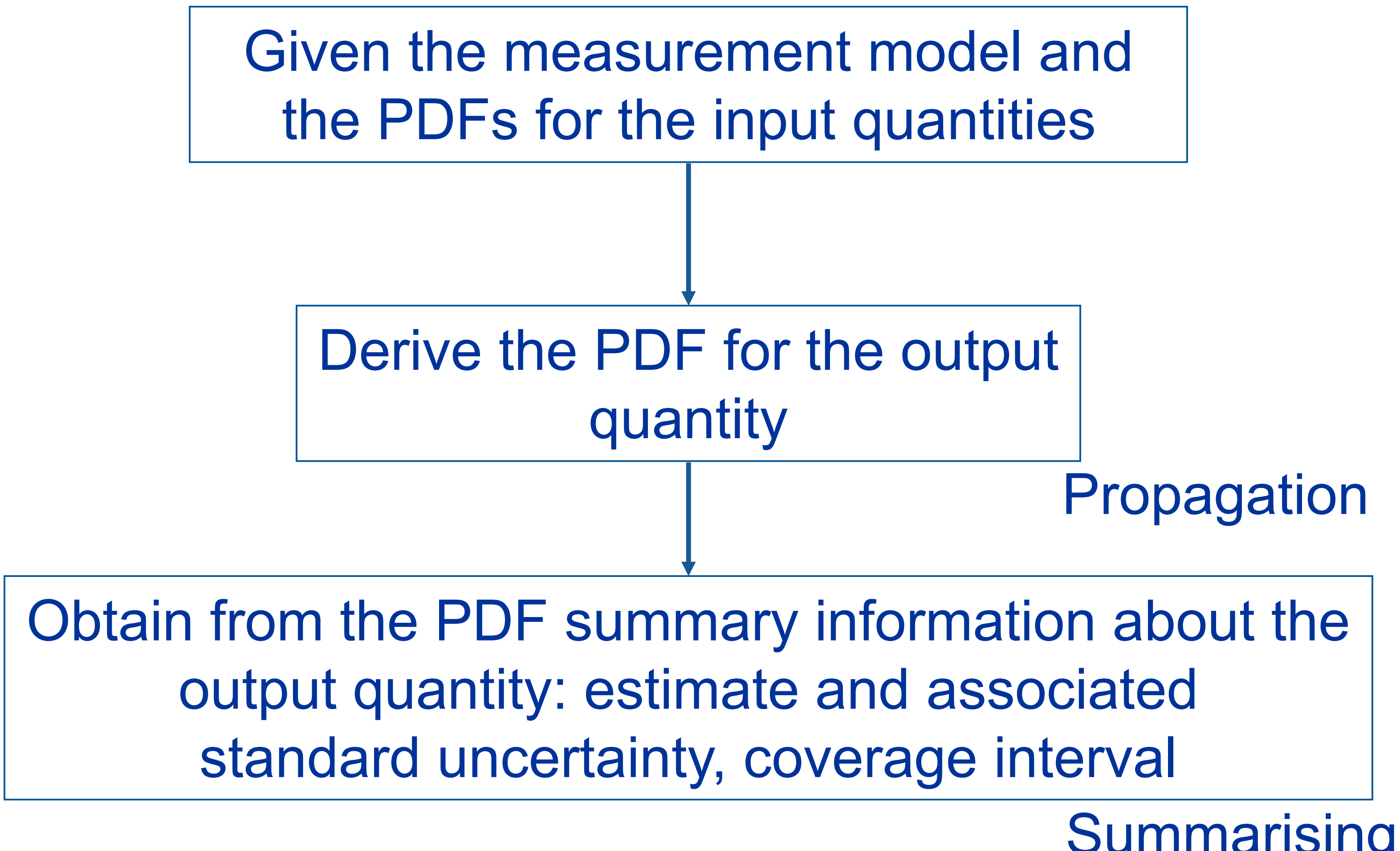
- Expresses the output quantity explicitly as a function of the input quantities
- Rule for delivering the output quantity given the input quantities



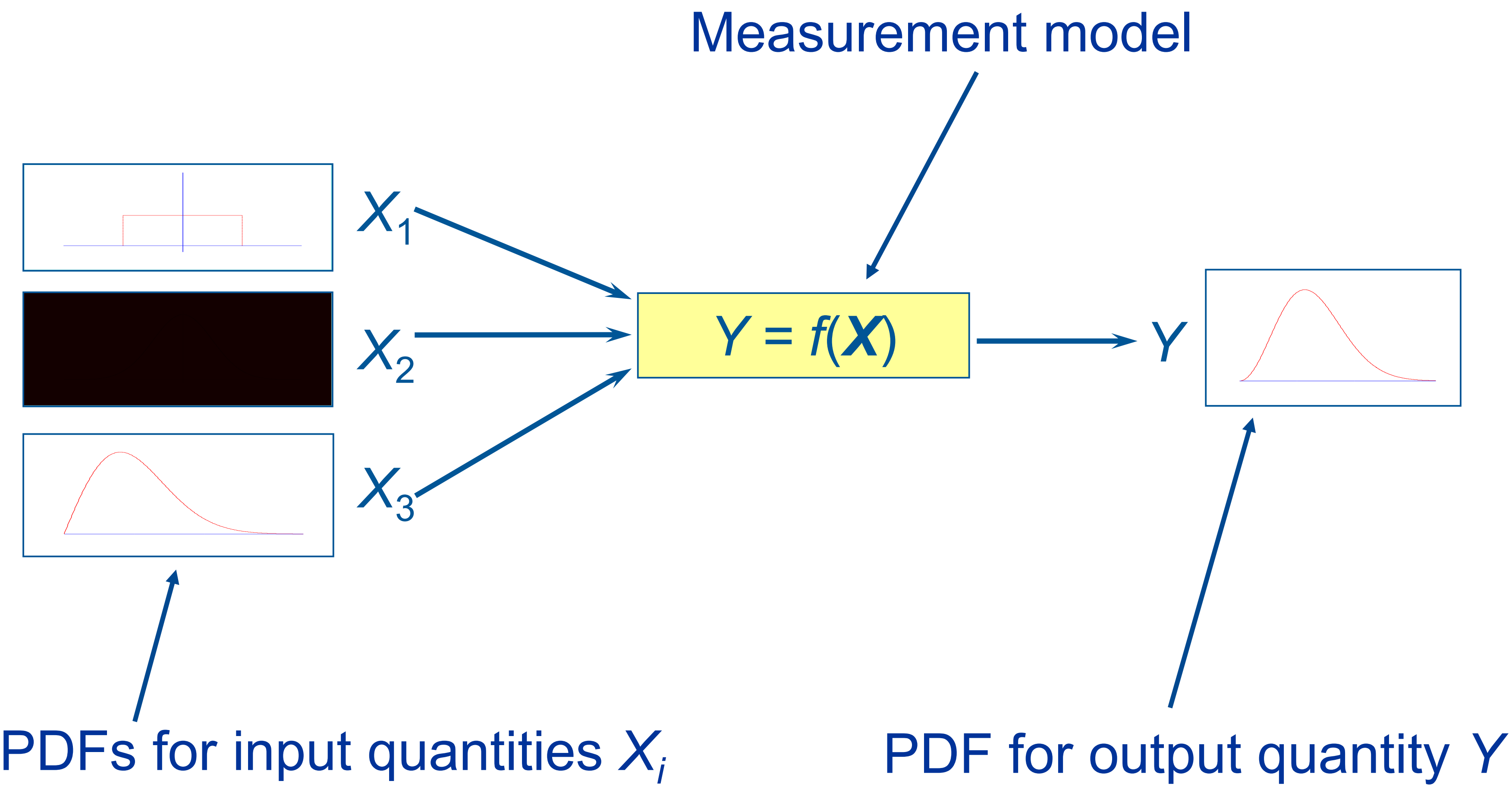
PDFs for input quantities

- Repeated indication values of the input quantity
  - Extract summary parameters (e.g., average, standard deviation associated with average)
  - Use PDF having these parameters
  - Type A evaluation
- Other (non-statistical) information relating to an input quantity
  - Use appropriate PDF based on historical data, supplier's statements, expert judgment, etc.
  - Type B evaluation

Calculation stage



Propagation of distributions



Obtaining summary information

- Expectation of Y

  $\rightarrow$ estimate y of Y

- Standard deviation of Y

  $\rightarrow$ standard uncertainty $u(y)$ associated with y

- Probability density function for Y

  $\rightarrow$ coverage interval corresponding to coverage probability p

Approaches to uncertainty evaluation

National Physical Laboratory

- GUM uncertainty framework
- "Other analytical or numerical methods": GUM-compliant

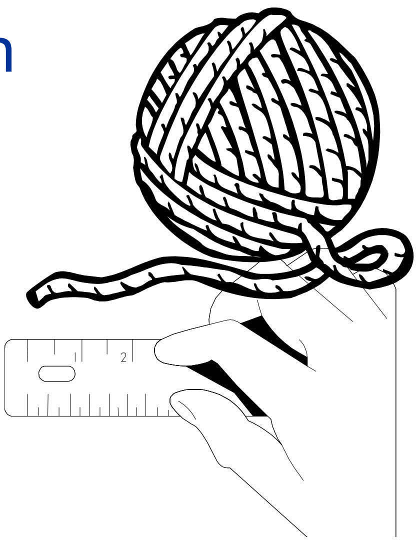
Why these stages?

National Physical Laboratory

- All metrological decisions in formulation stage
  - Measurement model
  - PDFs for the input quantities
- Calculation stage is then a defined problem
  - Mathematical
  - Statistical
  - Computational

Example of formulation: How long is a piece of string?

- Task: measure the length of a piece of string
- Measurement is made with a steel tape
- Output quantity (measurand): string length
- Input quantities?
- Measurement model?



Model and input quantities

String length = Measured string length (1)

+ Steel tape length correction (2)

+ String length correction (3)

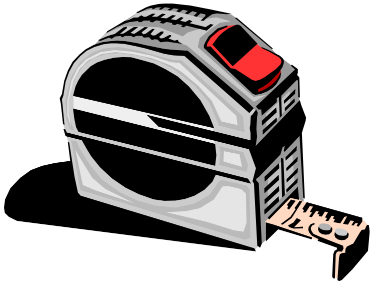
+ Measurement procedure correction (4)

(1) Measured string length

Measured string length = Average of a number of repeated indication values

(2) Steel tape length correction

National Physical Laboratory



Steel tape length correction =

Deviation due to tape calibration imperfections

\+ Deviation due to stretching of tape

\+ Deviation due to bending of tape

\+ Deviation due to tape thermal expansion

(3) String length correction

String length correction =

Deviation due to string departing from straight line

\+ Deviation as a result of shrinking of string

(4) Measurement procedure correction

Measurement procedure correction =

Deviation due to inability to align ends of steel tape and string due to fraying of the string ends

\+ Deviation due to steel tape and string not parallel

\+ Deviation due to interpolation of graduations on the steel tape to provide an indication value

After model in place

- Characterise quantity representing each term in the measurement model by a PDF

- Examples:

  Repeated indication values: use *t*-distribution

  Deviation due to interpolation of graduations: use rectangular distribution

GUM uncertainty framework

- Propagates estimates and standard uncertainties through measurement model
  - Law of propagation of uncertainty (LPU)
  - Only makes use of expectations and standard deviations of input quantities (and covariances and degrees of freedom where appropriate)

- Characterises the output quantity by a Gaussian distribution (or *t*-distribution)
  - Central limit theorem (CLT)
  - Uses this distribution to form a coverage interval

GUM uncertainty framework for general measurement model

$$Y = f(\boldsymbol{X}) \quad \text{with} \quad X_1, \dots, X_N \text{ independent}$$

- Propagate estimates x_i through measurement model

$$y = f(\boldsymbol{x})$$

- Propagate uncertainties $u(x_i)$ through linearised measurement model

$$u^2(y) = \sum_{i=1}^{N} c_i^2 u^2(x_i), \qquad c_i = \left.\frac{\partial f}{\partial X_i}\right|_{\boldsymbol{X}=\boldsymbol{x}}$$

$$y \approx E(Y), \qquad u^2(y) \approx V(Y)$$

GUM uncertainty framework for general measurement model

$$Y = f(\boldsymbol{X}) \quad \text{with} \quad X_1, \ldots, X_N \text{ independent}$$

- Equivalent form

$$u^2(y) = \sum_{i=1}^{N} c_i^2 u^2(x_i) = \sum_{i=1}^{N} u_i^2(y)$$

- Here,

$$u_i(y) = |c_i| u(x_i)$$

quantifies the contribution of the *i*th input quantity to $u(y)$

Evaluating a coverage region

National Physical Laboratory

- Use Welch-Satterthwaite formula to evaluate effective degrees of freedom ν_{eff}
  assumes input quantities are independent

- For $\nu_{\text{eff}} = \infty$, characterise $(Y - y)/u(y)$ by the standard Gaussian distribution N(0, 1)

- For $\nu_{\text{eff}} < \infty$, characterise $(Y - y)/u(y)$ by the *t*-distribution with ν_{eff} degrees of freedom

Welch-Satterthwaite formula

The Welch-Satterthwaite formula

$$\frac{u^4(y)}{\nu_{\text{eff}}} = \sum_{i=1}^{N} \frac{u_i^4(y)}{\nu_i}$$

or

$$\nu_{\text{eff}} = \frac{u^4(y)}{\sum_{i=1}^{N} u_i^4(y)/\nu_i}$$

ν_i = degrees of freedom attached to $u(x_i)$

ν_{eff} = degrees of freedom attached to $u(y)$

Constructing an uncertainty budget

- Tabular format commonly used
- Spreadsheet-friendly

Uncertainty budget

Source	Unc. Value	Dist.	k_i	c_i	$u_i(y)$	ν_i or ν_{eff}
X_1						
X_2						
X_3						
X_4						
X_5						
$\vdots$						
X_N						
$u(y)$						
U						

Uncertainty budget

Source	Unc. Value	Dist.	k_i	c_i	$u_i(y)$	ν_i or ν_{eff}
X_1						
X_2						
X_3						
X_4						
X_5						
⋮						
X_N						
$u(y)$						
U						

Divisor

Sensitivity coefficient

Contribution from *i*th quantity

Degrees of freedom

Uncertainty budget

Source	Unc. Value	Dist.	k_i	c_i	$u_i(y)$	ν_i or ν_{eff}
X_1						
X_2						
X_3						
X_4						
X_5						
⋮						
X_N						
$u(y)$						
U						

Combined standard uncertainty

Expanded uncertainty

Uncertainty budget

- Divisor k_i
  - Converts the uncertainty value to a standard uncertainty (corresponding to a standard deviation)
- Sensitivity coefficient c_i
  - Converts the standard uncertainty to the units of the measurand

$$c_i = \frac{\partial f}{\partial X_i} \text{ evaluated at } X_j = x_j, j = 1, \ldots, N$$

- Contribution to combined standard uncertainty

$$u_i(y) = \frac{\text{Unc value for } i\text{th quantity}}{k_i} \times |c_i|$$

Uncertainty budget

- Combined standard uncertainty $u(y)$
  - Calculated using law of propagation of uncertainty

$$u(y) = \sqrt{u_1^2(y) + \cdots + u_N^2(y)}$$

- Coverage factor k (corresponding to coverage probability p)
  - Based on characterising output quantity by Gaussian or *t*-distribution
  - E.g., $k = 1.96$ for $p = 0.95$ and $\boldsymbol{\nu}_{\mathbf{eff}}$ large
- Expanded uncertainty U

$$U = ku(y)$$

Further concepts

- Alternative approaches to calculation stage
  - Analytical methods (for simple cases only)
  - Monte Carlo method (GUM Supplement 1)
- Validation of GUM uncertainty framework
  - Using Monte Carlo method
- Multivariate models
  - More than one output quantity (GUM Supplement 2)
  - Coverage regions

Summary

National Physical Laboratory

- Probabilistic basis for uncertainty evaluation
- Measurement model relating input and output quantities
- GUM uncertainty framework

Useful links

- GUM (and accompanying documents)

  www.bipm.org/en/publications/guides/gum.html
- A beginner's guide to uncertainty of measurement

  www.npl.co.uk/publications/a-beginners-guide-to-uncertainty-in-measurement
- UKAS document M3003

  www.ukas.com/download/publications/publications-relating-to-laboratory-accreditation/M3003_Ed3_final.pdf
- VIM: International Vocabulary of Metrology

  www.bipm.org/en/publications/guides/vim.html

Useful links

- NPL uncertainty guides

  www.npl.co.uk/publications/uncertainty-guide/

- Mathematics, Modelling & Simulation software

  www.npl.co.uk/science-technology/mathematics-modelling-and-simulation/products-and-services/software-downloads

NPL National Physical Laboratory

Department for Business, Energy & Industrial Strategy

FUNDED BY BEIS

The National Physical Laboratory is operated by NPL Management Ltd, a wholly-owned company of the Department for Business, Energy and Industrial Strategy BEIS).