

PV ARRAY MODEL

1. Introduction

This document describes the modeling of the RTDS PV array model.

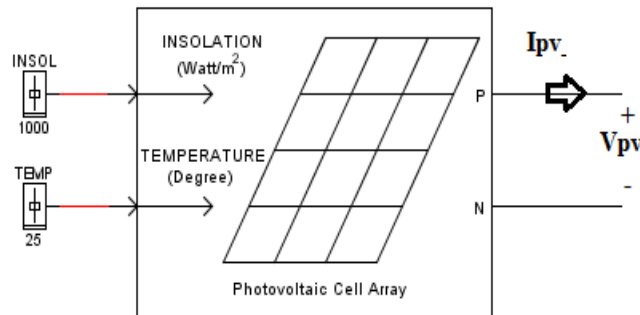


Figure 1 RTDS PV Array Model

The fundamental component of a PV array is the solar cell. Solar cells are manufactured using several types of semiconductor materials eg silicon and germanium; these semiconductor materials produce an electric charge when exposed to direct sunlight. Solar cells can be connected in series and/or parallel to form PV modules. A typical module will have 36/72 cells connected in series. The PV modules are then combined in series and parallel to form PV arrays. The combination of individual solar cells into PV arrays enables large values of voltages and currents to be obtained at the terminals of a PV array.

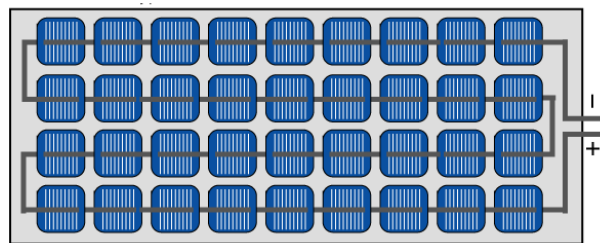


Figure 2 Series connected cells in a module [1]

The parameters that specify how the cells are connected to form arrays are the number of series connected cells in a module N_c , the number of parallel connected cells in a module N_{cp} , the number of series connected modules N_s and the number of parallel connected modules N_p .

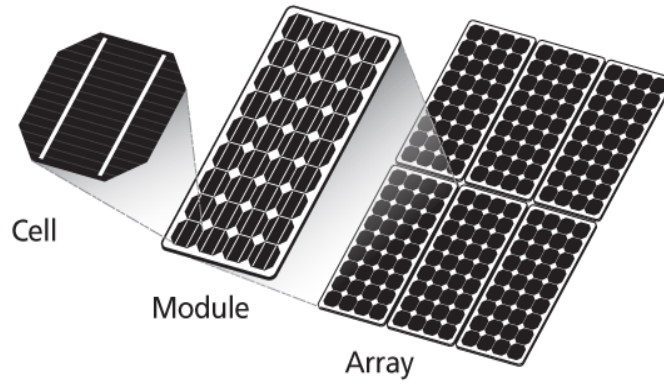
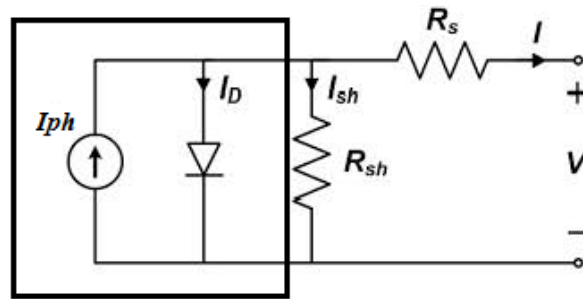


Figure 3 Cell-Module-Array [2]

2. Mathematical model of the solar cell

An ideal solar cell is electrically represented as a current source in parallel with a single diode, however practical solar cell models require additional elements to accurately represent their nonlinear current-voltage characteristics. A commonly used practical model extends the ideal solar cell by including a series and shunt resistance as shown in figure 3.



Ideal Solar Cell

Figure 4 Single diode, five parameter model

The current-voltage relationship of the single diode, five parameter model is given by:

$$I = I_{ph} - I_D - I_{sh} \quad (1)$$

$$I = I_{ph} - I_o \left(\exp \left(\frac{V + R_s I}{N_c a V_t} \right) - 1 \right) - \left(\frac{V + R_s I}{R_{sh}} \right) \quad (2)$$

The photocurrent I_{ph} is the current induced by the incidence of sunlight on the solar cell. The diode current I_D produces the I-V characteristic of the PV cell where I_o is the diode reverse saturation current. The diode ideality factor a is a measure of how closely the diode matches the ideal diode equation. The series resistance R_s is the sum of several structural resistances in the solar cell while the shunt resistance R_{sh} represents the leakage current of the semiconductor material and the

manufacturing process of the solar cell [2, 3]. V_t is the diode thermal voltage which is a constant defined at any given temperature T (in K) by:

$$V_t = \frac{kT}{q} \quad (3)$$

k is the Boltzmann constant ($1.3806503 \times 10^{-23} \text{J/K}$) and q is the magnitude of an electron charge ($1.602176 \times 10^{-19} \text{C}$).

The single, diode five parameter model can be further simplified by neglecting the effect of the shunt resistance (assuming a very high shunt resistance value $> 1000 \text{ ohm}$) as shown in figure 5.

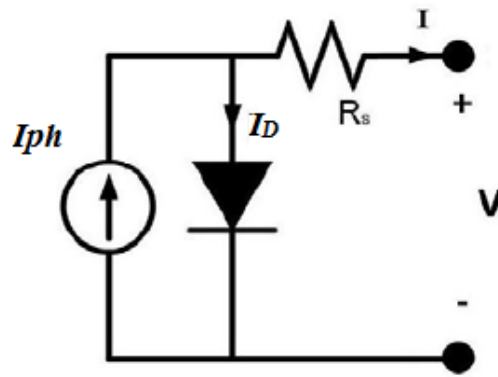


Figure 5 Single diode, four parameter model

The current-voltage relationship of the single diode, four parameter model is given by:

$$I = I_{ph} - I_o \left(\exp \left(\frac{V + R_s I}{N_c a V_t} \right) - 1 \right) \quad (7)$$

The RTDS PV array model includes the option to select either the single diode, four parameter model or the single diode, five parameter model.

The I-V characteristics define the operating point of the PV array for a given isolation and temperature as shown in figure 5. The curves range from the short circuit current ($I_{sc}, 0$) to the open circuit voltage ($0, V_{oc}$) with a knee point (I_m, V_m) defined as the maximum power point where the PV array generates maximum electrical power P_{max} .

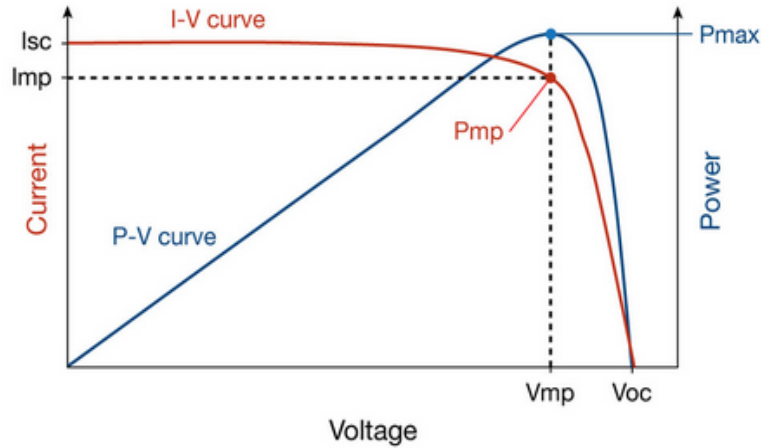


Figure 6 I-V and P-V curve of a solar cell

The I-V characteristic of the solar cell depends on the variation of the solar parameters with the given insolation G (solar intensity W/m^2) and temperature T (K) as described by (4) - (6) where k_i is the short circuit current temperature coefficient ($\%/^{\circ}\text{C}$) and E_g is the energy gap (eV) of the selected solar cell semiconductor material.

$$I_{ph} = \frac{G}{G_{ref}} * (I_{ph_{ref}} + k_i(T - T_{ref})) \quad (4)$$

$$I_o = I_{oref} \left(\frac{T}{T_{ref}} \right)^3 \exp\left(\frac{E_g}{aV_t} \left(1 - \frac{T_{ref}}{T} \right) \right) \quad (5)$$

$$a = a_{ref} \left(\frac{T}{T_{ref}} \right) \quad (6)$$

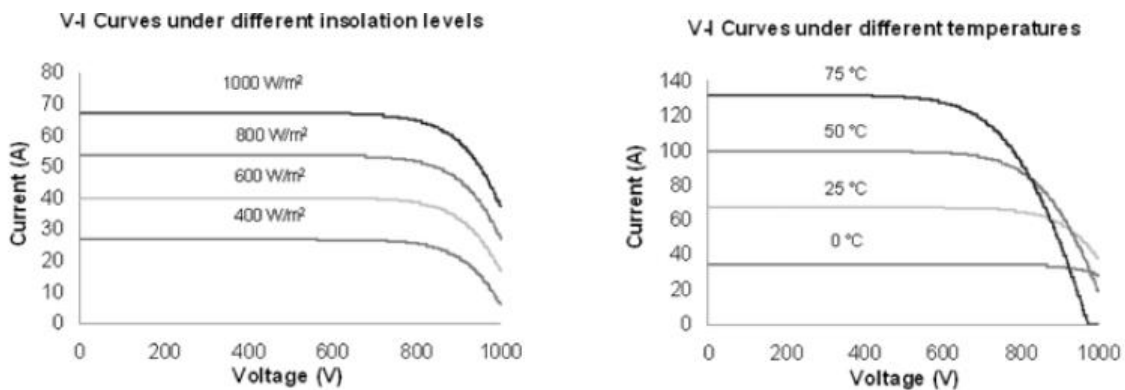


Figure 7 Variation of I-V curves with different G and T values.

Generally the parameters ($I_{ph_{ref}}, I_{oref}, a_{ref}, R_s, R_{sh}$) are not made available by the PV manufacturers and have to be estimated from available data ($I_{sc_{ref}}, V_{oc_{ref}}, I_{m_{ref}}, V_{m_{ref}}, R_{so}, R_{sho}$) which are specified in the PV data sheets at stc $G_{ref} = 1000 \frac{W}{m^2}, T_{ref} = 25^{\circ}$.

The RTDS PV array model includes the option to estimate the unknown solar cell parameters using either an analytical method or an iterative method.

3. Analytical method for solar cell parameter estimation

For the single diode, four parameter model, the four unknown parameters are estimated using the following analytical expressions [3].

$$I_{ph_{ref}} = I_{sc_{ref}} \quad (8)$$

$$a_{ref} = \frac{q(2V_{m_{ref}} - V_{oc_{ref}})}{N_c k T_{ref} \left[\frac{I_{sc_{ref}}}{I_{sc_{ref}} - I_{m_{ref}}} + \ln \left(1 - \frac{I_{m_{ref}}}{I_{sc_{ref}}} \right) \right]} \quad (9)$$

$$I_{oref} = \frac{I_{sc_{ref}}}{\left(\exp \left(\frac{V_{oc_{ref}}}{N_c a_{ref} V_t} \right) - 1 \right)} \quad (10)$$

$$R_s = \frac{N_c a_{ref} V_t \ln \left(1 - \frac{I_{m_{ref}}}{I_{sc_{ref}}} \right) + (V_{oc_{ref}} - V_{m_{ref}})}{I_{m_{ref}}} \quad (11)$$

For the single diode, five parameter model shown in figure 3, the five unknown parameters are estimated using the following analytical expressions [4].

$$R_{sh} = R_{sho} \quad (12)$$

$$a_{ref} = \frac{V_{m_{ref}} + I_{m_{ref}} R_{so} - V_{oc_{ref}}}{V_t \left[\ln \left(I_{sc_{ref}} - \frac{V_{m_{ref}}}{R_{sho}} - I_{m_{ref}} \right) - \ln \left(I_{sc_{ref}} - \frac{V_{oc_{ref}}}{R_{sho}} \right) - \left(\frac{I_{m_{ref}}}{I_{sc_{ref}} - \frac{V_{oc_{ref}}}{R_{sho}}} \right) \right]} \quad (13)$$

$$I_{oref} = \frac{I_{scref} - \frac{V_{ocref}}{R_{sh}}}{\exp\left(\frac{V_{ocref}}{N_c a_{ref} V_t}\right)} \quad (14)$$

$$R_s = R_{so} - \left(\frac{\frac{a_{ref} V_t}{I_{oref}}}{\exp\left(\frac{V_{ocref}}{N_c a_{ref} V_t}\right)} \right) \quad (15)$$

$$I_{phref} = I_{scref} \left(1 + \frac{R_s}{R_{sh}}\right) + I_{oref} \left(\exp\left(\frac{I_{scref} R_s}{a_{ref} V_t}\right) - 1\right) \quad (16)$$

4. Iterative method for solar cell parameter estimation

The RTDS PV array model uses the Newtho-Raphson algorithm which converges to the roots of a real valued function, within a desirable tolerance, using successive approximations from a given initial point [5]. The details of the Newtho-Raphson algorithm for the solar array model is given in the Appendix. The flowchart of the Newtho-Raphson algorithm is shown in figure 8. The initial starting point is estimated using the analytical expressions given by equations (8)-(16). The maximum number of iterations is fixed at 100. The tolerance value and maximum value are fixed at $1e^{-10}$ and $10e^3$ respectively.

The reference solar cell parameters obtained from the analytical or iterative methods are then used in equations (4)-(6) to obtain the total PV array current given by:

For the single diode, four parameter model:

$$I_{pv} = N_{cp} N_p I_{ph} - N_{cp} N_p I_o \left(\exp\left(\frac{V_{pv} + \frac{N_s}{N_p N_{cp}} R_s I_{pv}}{N_s N_{cs} a V_t}\right) - 1 \right)$$

For the single diode, five parameter model:

$$I_{pv} = N_{cp} N_p I_{ph} - N_{cp} N_p I_o \left(\exp\left(\frac{V_{pv} + \frac{N_s}{N_p N_{cp}} R_s I_{pv}}{N_s N_{cs} a V_t}\right) - 1 \right) - \frac{V_{pv} + \frac{N_s}{N_p N_{cp}} R_s I_{pv}}{\frac{N_s}{N_p N_{cp}} R_{sh} \frac{G}{G_{ref}}}$$

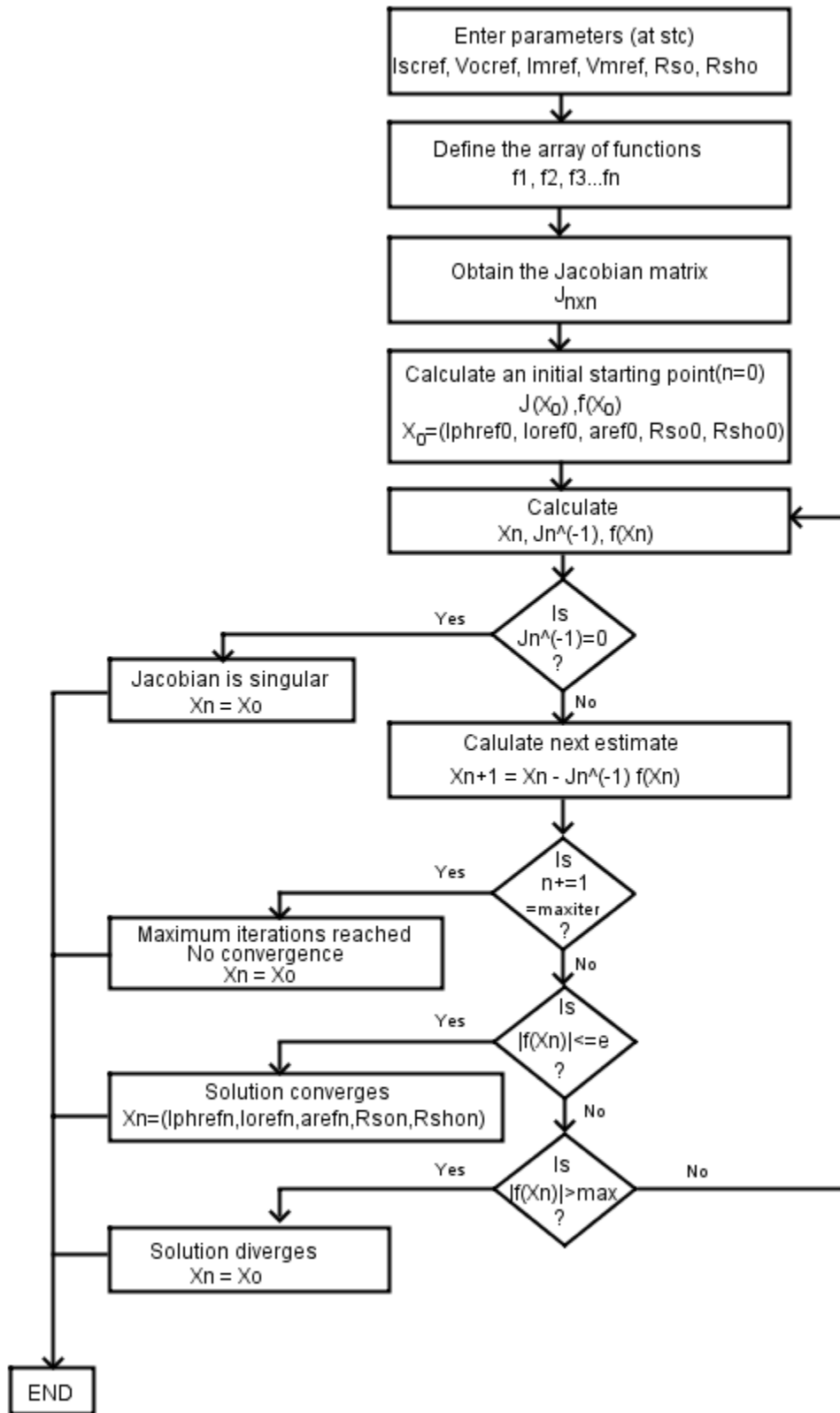
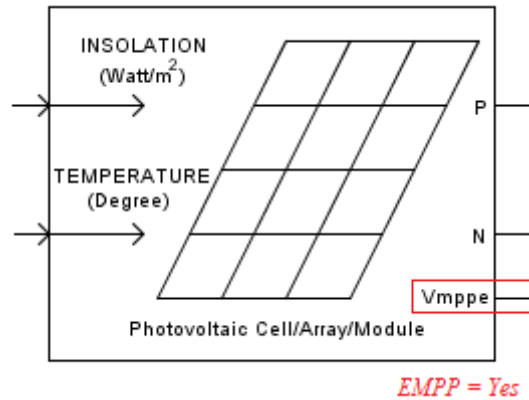


Figure 8 Flowchart of Newton-Raphson method for solar cell parameter estimation

5. Maximum power point tracking

The RTDS PV model includes the option to estimate the maximum power point for a given insolation and temperature using two analytical approximations: 1. Lambert function approximation 2. Fractional open circuit voltage approximation.



SIGNAL MONITORING			SIGNAL NAMES			PROCESSOR ASSIGNMENT		
CONFIGURATION			MODULE DATA AND CONFIGURATION			ARRAY CONF		
Name	Description	Value						
Name	Component name	MLE255HD1						
Model_Type	Select solar cell model	Single Diode Four Parameter ...						
calcPara	Select calculation of solar cell parameters	Analytical Method						
EMPP	Enable estimation of maximum power point	Yes						
MPPT	Select MPP estimation technique	Fractional open circuit voltage						
EnbShd	Enable shading effect	Lambert function approximation						
Mon	Enable monitoring of PV Parameters	Fractional open circuit voltage						
prtyp	Solve Model on card type:	GPC/PB5						

Figure 9 Enable maximum power point estimation

5.1 Fractional open circuit voltage

The maximum power point is estimated from the open circuit voltage using the following expression (where Kv is the open circuit voltage temperature coefficient in %/degC).

$$V_{oc} = V_{ocref} + (Kv/100) * (T - T_{ref}) + aVar * \log(G/G_{ref})$$

$$V_{mp} = (80\%)V_{oc}$$

5.2 Lambert function approximation

The expression for the maximum power point using the Lambert function is given by [6]:

Single diode, five parameter model:

$$V_{mp} = V_{im} - \left(I_{im} - \frac{V_{im}}{R_{sh}} \right) R_s$$

Single diode, four parameter model:

$$V_{mp} = V_{im} - I_{im} R_s$$

Where:

$$V_{im} = N_s N_{cs} a V_t \left(\text{lambertW} \left(\frac{I_{ph} \exp(1)}{I_o} \right) - 1 \right)$$

$$I_{im} = \frac{V_{im} I_o}{N_s N_{cs} a V_t} \exp \left(\frac{V_{im}}{N_s N_{cs} a V_t} \right)$$

6. Modeling Shading Effects

A significant impact on the operation of PV modules is shading caused by cloud cover, trees or buildings. Shading creates mismatch losses in PV modules because the output of the entire PV module is determined by the shaded solar modules with the lowest output [1]. Typically PV modules are equipped with bypass diodes which protect the PV module from heating caused from current mismatches during shading conditions and blocking diodes which protect the unshaded PV array strings from reverse currents caused by the shaded strings. The addition of the bypass diodes creates multiple maximum power point peaks due to the different insolation levels. Multiple maximum points can “confuse” traditional hill climbing MPPT schemes as they can easily track and settle at a local maxima which reduces the available power output from the PV array.

The RTDS PV model includes an option to include shading effects as shown in figure 10 and 11. When the enable shading parameter is selected the user has the option to include bypass diodes and/or blocking diodes in the PV array model. This option allows users to design and/or test a maximum power point tracking scheme that can track the global maxima of the PV array under shading conditions.

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PROCESSOR ASSIGNMENT		SHADING EFFECTS	
SIGNAL MONITORING		SIGNAL NAMES	
CONFIGURATION		MODULE DATA AND CONFIGURATION	
Name	Description	Value	
Name	Component name	MLE255HD1	
Model_Type	Select solar cell model	Single Diode Four Parameter ... ▼	
calcPara	Select calculation of solar cell parameters	Analytical Method ▼	
EMPP	Enable estimation of maximum power point	Yes ▼	
MPPT	Select MPP estimation technique	Fractional open circuit voltae ▼	
EnbShd	Enable shading effect	Yes ▼	
Mon	Enable monitoring of PV Parameters	Yes ▼	
prtyp	Solve Model on card type:	GPC/PB5 ▼	

PROCESSOR ASSIGNMENT		SHADING EFFECTS	
CONFIGURATION		MODULE DATA AND CONFIGURATION	
ARRAY CONFIGURATION		ARRAY CONFIGURATION	
Name	Description	Value	
bypDs	Include bypass diodes (one bypass diode per module)	No ▼	
blkDd	Include blocking diode (one blocking diode per PV string)	No ▼	
ShdMds	Number of shaded modules in the PV string (ShdMds<=Ns)	1	
ShdMdstrgs	Number of shaded PV strings (shdMdstrgs<=Np)	1	
ShdStrtOpt	Shading strength control option	CONSTANT ▼	
ShdStrt	Shading strength (Insolation of shaded modules)	500	
ShdStrtCC	Shading strength CC Signal name	INS	

Figure 10 Enable shading effect in PV model

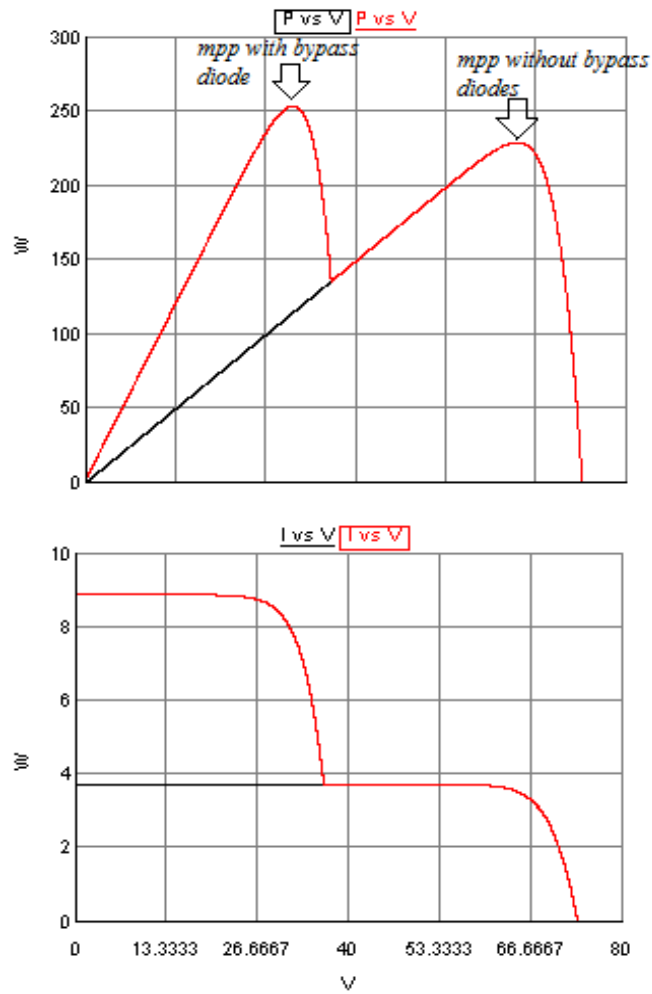
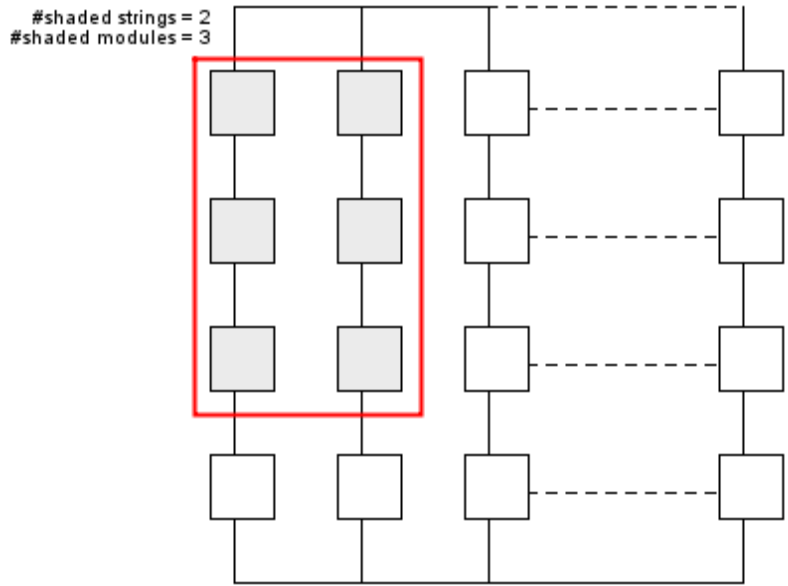


Figure 11 I-V and P-V curves with (red) and without (black) bypass diodes

References

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- [2] [Online]. Available: <http://www.samlexsolar.com/learning-center/solar-cell-module-array.aspx>.

- [3] D. S. Chan and J. C. Phang, "Analytical Methods for the Extraction of Solar-Cell Single-and Double-Diode Model Parameters from I-V Characteristics," *IEEE Transactions on Electronic Devices*, vol. 34, no. 2, pp. 286-293, February 1987.

- [4] R. Khezzar, M. Zereg and A. Khezzar, "Comparative Study of Mathematical Methods for Parameters Calculation of Current-Voltage Characteristics of Photovoltaic Module," in *International Conference on Electrical and Electronics Engineering 2009. ELECO 2009*, Bursa, 2009.

- [5] Gilberto, "Solution of non linear equations," [Online]. Available: http://ocw.usu.edu/Civil_and_Environmental_Engineering/Numerical_Methods_in_Civil_Engineering/NonLinearEquationsMatlab.pdf.

- [6] G.Farivar, B.Asaei, S.Mehrnam, "An Analytical Solution for Tracking Photovoltaic Module MPP," *IEEE Journal of Photovoltaics*, vol.3, no.3, July 2013.

Appendix

Newthon-Raphson algorithm

For a real-valued function $f(x) = 0$, the solution x_{i+1} is found using tangent line approximations from an initial point x_0 .

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad (17)$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad (18)$$

The algorithm converges to a solution when:

$$|f(x_{i+1})| < \varepsilon \quad (19)$$

ε is the desired tolerance or error value.

For a system with n non-linear equations and n unknowns, the Newthon –Raphson algorithm requires the array of functions as well as the array of the derivatives of the functions known as the Jacobian matrix to solve for the roots $(x_{i+1}1, x_{i+1}2, x_{i+1}3, x_{i+1}4, x_{i+1}5, \dots x_{i+1}n)$ [6].

$$\begin{aligned} f_1(x_1, x_2, x_3, x_4, x_5, \dots, x_n) &= 0 \\ f_2(x_1, x_2, x_3, x_4, x_5, \dots, x_n) &= 0 \\ &\vdots \\ &\vdots \\ &\vdots \\ f_n(x_1, x_2, x_3, x_4, x_5, \dots, x_n) &= 0 \end{aligned} \quad (20)$$

The array of functions can be written as:

$$f(\mathbf{X}) = \mathbf{0} \quad (21)$$

$$f(\mathbf{X}) = \begin{bmatrix} f_1(\mathbf{X}) \\ f_2(\mathbf{X}) \\ \vdots \\ f_n(\mathbf{X}) \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2 \dots x_n) \\ f_2(x_1, x_2 \dots x_n) \\ \vdots \\ f_n(x_1, x_2 \dots x_n) \end{bmatrix} = \mathbf{0}; \quad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad (22)$$

The Jacobian matrix is obtained from the partial derivative of the functions:

$$J = \frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x_1, x_2, \dots, x_n)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad (23)$$

The Newton-Raphson estimates are calculated using the expression:

$$X_{n+1} = X_n - J^{-1}f(X_n) = X_n - \Delta X_n \quad (24)$$

The convergence criteria for the solution of the non-linear equations is given as:

$$|f(X_{n+1})| < \varepsilon \quad (25)$$

6.1 Defining the array of functions

The non-linear functions for the four and five parameter solar cell models are derived using (2) and (7) as well as points on the I-V and P-V curves at standard test conditions.

A. For the single diode, five parameter model with unknowns ($I_{ph_{ref}}$, I_{oref} , a_{ref} , R_s , R_{sh}):

$$I - I_{ph} + I_o \left(\exp\left(\frac{V + R_s I}{N_c a_{ref} V_t}\right) - 1 \right) + \left(\frac{V + R_s I}{R_{sh}}\right) = 0 \quad (26)$$

- At $I = 0$, $V = V_{oc_{ref}}$:

$$I_{ph} = I_o \left(\exp\left(\frac{V_{oc_{ref}}}{N_c a_{ref} V_t}\right) - 1 \right) + \frac{V_{oc_{ref}}}{R_{sh}} \quad (27)$$

- At $I = I_{sc_{ref}}$, $V = 0$ and using (27) (f1):

$$I_o \left(\exp\left(\frac{V_{oc_{ref}}}{N_c a_{ref} V_t}\right) - \exp\left(\frac{I_{sc_{ref}} R_s}{N_c a_{ref} V_t}\right) \right) - I_{sc_{ref}} \left(1 + \frac{R_s}{R_{sh}} \right) + \frac{V_{oc_{ref}}}{R_{sh}} = 0 \quad (28)$$

- At $I = I_{m_{ref}}$, $V = V_{m_{ref}}$ and using (27) (f2):

$$I_{m_{ref}} + I_o \left(\exp \left(\frac{V_{m_{ref}} + R_s I_{m_{ref}}}{N_c a_{ref} V_t} \right) - \exp \left(\frac{V_{oc_{ref}}}{N_c a_{ref} V_t} \right) \right) + \left(\frac{V_{m_{ref}} + R_s I_{m_{ref}} - V_{oc_{ref}}}{R_{sh}} \right) = 0 \quad (29)$$

- At $\frac{dI}{dV} |_{V=V_{oc_{ref}}} = \frac{-1}{R_{so}}$ (f3):

$$I_o \left(\frac{1}{N_c a_{ref} V_t} \exp \left(\frac{V_{oc_{ref}}}{N_c a_{ref} V_t} \right) \left(1 - \frac{R_s}{R_{so}} \right) \right) + \frac{1}{R_{sh}} \left(1 - \frac{R_s}{R_{so}} \right) - \frac{1}{R_{so}} = 0 \quad (30)$$

- At $\frac{dI}{dV} |_{I=I_{sc_{ref}}} = \frac{-1}{R_{sho}}$ (f4):

$$I_o \left(\frac{1}{N_c a_{ref} V_t} \exp \left(\frac{I_{sc_{ref}}}{N_c a_{ref} V_t} \right) \left(1 - \frac{R_s}{R_{sho}} \right) \right) + \frac{1}{R_{sh}} \left(1 - \frac{R_s}{R_{sho}} \right) - \frac{1}{R_{sho}} = 0 \quad (31)$$

The Jacobian matrix is obtained from the partial derivatives of the functions given in equations (29)-(32):

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial I_o} & \frac{\partial f_1}{\partial a_{ref}} & \frac{\partial f_1}{\partial R_s} & \frac{\partial f_1}{\partial R_{sh}} \\ \frac{\partial f_2}{\partial I_o} & \frac{\partial f_2}{\partial a_{ref}} & \frac{\partial f_2}{\partial R_s} & \frac{\partial f_2}{\partial R_{sh}} \\ \frac{\partial f_3}{\partial I_o} & \frac{\partial f_3}{\partial a_{ref}} & \frac{\partial f_3}{\partial R_s} & \frac{\partial f_3}{\partial R_{sh}} \\ \frac{\partial f_4}{\partial I_o} & \frac{\partial f_4}{\partial a_{ref}} & \frac{\partial f_4}{\partial R_s} & \frac{\partial f_4}{\partial R_{sh}} \end{bmatrix} \quad (32)$$

The Newthton-Raphson solution for $(I_o=I_{oref}, a_{ref}, R_s, R_{sh})$ is then used to calculate $(I_{ph}=I_{ph_{ref}})$ using equation (27).

B. For the single diode, four parameter model with unknowns $(I_{ph_{ref}}, I_{oref}, a_{ref}, R_s)$:

$$I - I_{ph} + I_o \left(\exp \left(\frac{V + R_s I}{N_c a V_t} \right) - 1 \right) = 0 \quad (33)$$

- At $I = 0, V = V_{oc_{ref}}$:

$$I_{ph} = I_o \left(\exp \left(\frac{V_{oc_{ref}}}{N_c a_{ref} V_t} \right) - 1 \right) \quad (34)$$

- At $I = I_{sc_{ref}}, V = 0$ and using (34) (f1):

$$I_o \left(\exp \left(\frac{V_{oc_{ref}}}{N_c a_{ref} V_t} \right) - \exp \left(\frac{I_{sc_{ref}} R_s}{N_c a_{ref} V_t} \right) \right) - I_{sc_{ref}} = 0 \quad (35)$$

- At $I = I_{m_{ref}}, V = V_{m_{ref}}$ and using (34) (f2):

$$I_o \left(\exp \left(\frac{V_{m_{ref}} + R_s I_{m_{ref}}}{N_c a_{ref} V_t} \right) - \exp \left(\frac{V_{oc_{ref}}}{N_c a_{ref} V_t} \right) \right) - I_{m_{ref}} = 0 \quad (36)$$

- At $\frac{dP}{dV} |_{I=I_{m_{ref}}, V=V_{m_{ref}}} (f3)$:

$$I_o \left(\frac{1}{N_c a_{ref} V_t} \exp \left(\frac{V_{m_{ref}} + R_s I_{m_{ref}}}{N_c a_{ref} V_t} \right) \left(1 - \frac{I_{m_{ref}}}{V_{m_{ref}}} R_s \right) \right) - \frac{I_{m_{ref}}}{V_{m_{ref}}} = 0 \quad (37)$$

The Jacobian matrix is obtained from the partial derivatives of the functions given in equations (35) – (37):

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial I_0} & \frac{\partial f_1}{\partial a_{ref}} & \frac{\partial f_1}{\partial R_s} \\ \frac{\partial f_2}{\partial I_0} & \frac{\partial f_2}{\partial a_{ref}} & \frac{\partial f_2}{\partial R_s} \\ \frac{\partial f_3}{\partial I_0} & \frac{\partial f_3}{\partial a_{ref}} & \frac{\partial f_3}{\partial R_s} \end{bmatrix} \quad (38)$$

To reduce the modeling complexity, the size of the Jacobian is reduced by using R_s given by equation (15). The Newton-Raphson solution for $(I_o = I_{oref}, a_{ref})$ is then used to calculate $(I_{ph} = I_{ph_{ref}})$ using equation (34).

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial I_0} & \frac{\partial f_1}{\partial a_{ref}} \\ \frac{\partial f_2}{\partial I_0} & \frac{\partial f_2}{\partial a_{ref}} \end{bmatrix} \quad (39)$$